Closing Thur: 13.2
Closing Fri: 13.3
Closing Tues: 13.4
Exam 1 is Tues (Oct. 23)
covers 12.1-12.6, 13.1-13.4

13.3 Measurements on Curves in 3D

Today: *Unit Tangent and Normal, Arc Length, and Curvature*

Entry Task

Consider $\boldsymbol{r}(t) = \langle t, t^2, 5\rangle$.

Find the unit tangent vector **T**(t).

$$\vec{r}\,'(t) = \langle 1, 2t, 0\rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1+4t^2+0}$$

$$\boxed{\vec{T}(t) = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}}, 0\right\rangle}$$

NOTE

$$\vec{T}\cdot\vec{T}\,' = -\frac{4t}{(1+4t^2)^2} + \frac{4t}{(1+4t^2)^2} = 0 \checkmark$$

Thm: $\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.

Proof: Since $\boldsymbol{T}\cdot\boldsymbol{T} = |\boldsymbol{T}|^2 = 1$,
we differentiate both sides to get

$$\boldsymbol{T}'\cdot\boldsymbol{T} + \boldsymbol{T}\cdot\boldsymbol{T}' = 0.$$

So $2\boldsymbol{T}\cdot\boldsymbol{T}' = 0$.
Thus, $\boldsymbol{T}\cdot\boldsymbol{T}' = 0$. (QED)

ASIDE

$$\frac{d}{dt}\left((1+4t^2)^{-1/2}\right) = -\frac{1}{2}(1+4t^2)^{-3/2}\cdot 8t = \frac{-4t}{(1+4t^2)^{3/2}}$$

$$\frac{d}{dt}\left(\frac{2t}{(1+4t^2)^{1/2}}\right) = \left(\frac{\sqrt{1+4t^2}\cdot 2 - 2t\,\frac{8t}{2\sqrt{1+4t^2}}}{1+4t^2}\right)\frac{\sqrt{1+4t^2}}{\sqrt{1+4t^2}}$$

SIMPLIFY

$$= \frac{(1+4t^2)\cdot 2 - 8t^2}{(1+4t^2)^{3/2}}$$

$$= \frac{2}{(1+4t^2)^{3/2}}$$

$$\Rightarrow \boxed{\vec{T}\,'(t) = \left\langle -\frac{4t}{(1+4t^2)^{3/2}}, \frac{2}{(1+4t^2)^{3/2}}, 0\right\rangle}$$

13.3 Measurements on Curves in 3D

$x'(t) = 3$
$y'(t) = 2$
$z'(t) = 5$

Distance Traveled on a Curve

The dist. traveled along a curve from
$t = a$ to $t = b$ is

$$\boxed{\int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt}$$

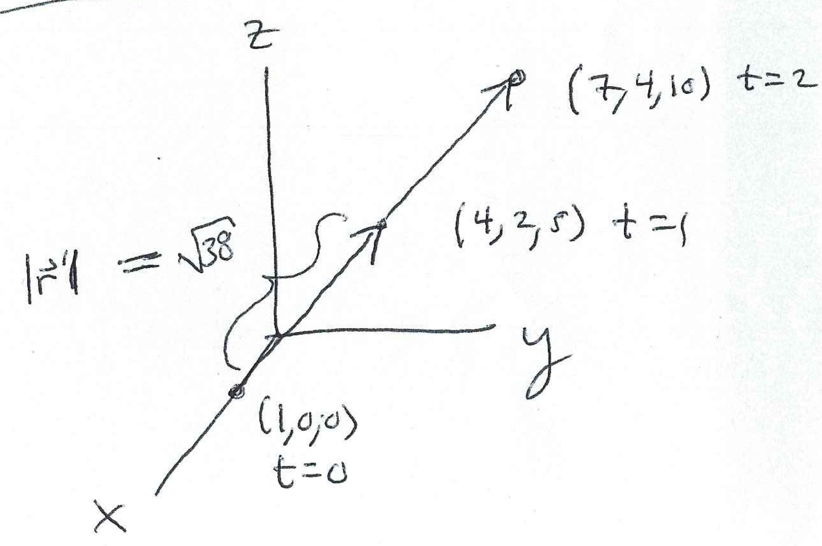
Note: 2D is same without the $z'(t)$.
We derived this in Math 125.

$$\text{DISTANCE} = \int_0^2 \sqrt{(3)^2 + (2)^2 + (5)^2}\,dt$$
$$= \int_0^2 \sqrt{9 + 4 + 25}\,dt$$
$$= \int_0^2 \sqrt{38}\,dt$$
$$= \sqrt{38}\,t \Big|_0^2$$
$$= 2\sqrt{38}$$

Examples:

1. Find the length of the curve
$$\mathbf{r}(t) = \langle 1 + 3t, 2t, 5t \rangle$$
from $t = 0$ to $t = 2$.

NOTE: LINEAR MOTION



13.3 Measurements on Curves in 3D

Goal: distance/arc length, unit tangent, unit normal, curvature.

Entry Task: Find the length of the curve $\mathbf{r}(t) = \langle \cos(2t), \sin(2t), 2\ln(\cos(t)) \rangle$

$\boxed{0 \text{ to } \pi/3}$

$x'(t) = -2\sin(2t)$

$y'(t) = 2\cos(2t)$

$z'(t) = 2 \frac{1}{\cos(t)} \sin(t) = 2\tan(t)$

$= \int_0^{\pi/3} \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 4\tan^2(t)}\, dt$

$= \int_0^{\pi/3} \sqrt{4 + 4\tan^2(t)}\, dt$

$= 2 \int_0^{\pi/3} \sqrt{1 + \tan^2(t)}\, dt$

$= 2 \int_0^{\pi/3} \sqrt{\sec^2(t)}\, dt$

$= 2 \int_0^{\pi/3} \sec(t)\, dt$

$= 2 \ln|\sec(t) + \tan(t)| \Big|_0^{\pi/3}$

$= 2\ln|\underbrace{\sec(\pi/3)}_{2} + \underbrace{\tan(\pi/3)}_{\sqrt{3}}| - 2\ln|\underbrace{\sec(0)}_{1} + \underbrace{\tan(0)}_{0}|$

$= \boxed{2\ln(2+\sqrt{3})} \approx 2.63$

Distance Traveled on a Curve

The dist. traveled along a curve from $t = a$ to $t = b$ is

$$\boxed{\int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt}$$

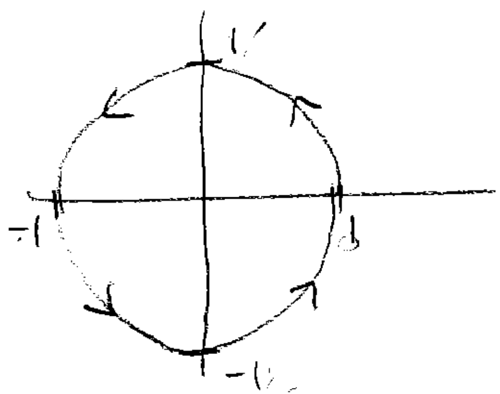
Note: 2D is same without the $z'(t)$. We derived this in Math 125.

If the curve is "traversed once" we call this **arc length**.

Example: x = cos(t), y = sin(t)

(a) Find the distance traveled by this object from t = 0 to t = 6π.

(b) Find the arc length of the path over which this object is traveling.



$$\int_0^{6\pi} \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$

$$\int_0^{6\pi} \sqrt{1}\, dt$$

$$\int_0^{6\pi} 1\, dt = t \Big|_0^{6\pi}$$

$$= \boxed{6\pi} = \text{DISTANCE TRAVELED}$$

TRAVERSED ONCE FROM $t=0$ to $t=2\pi$

$$\int_0^{2\pi} \sqrt{1}\, dt = \boxed{2\pi} = \text{ARC LENGTH}$$

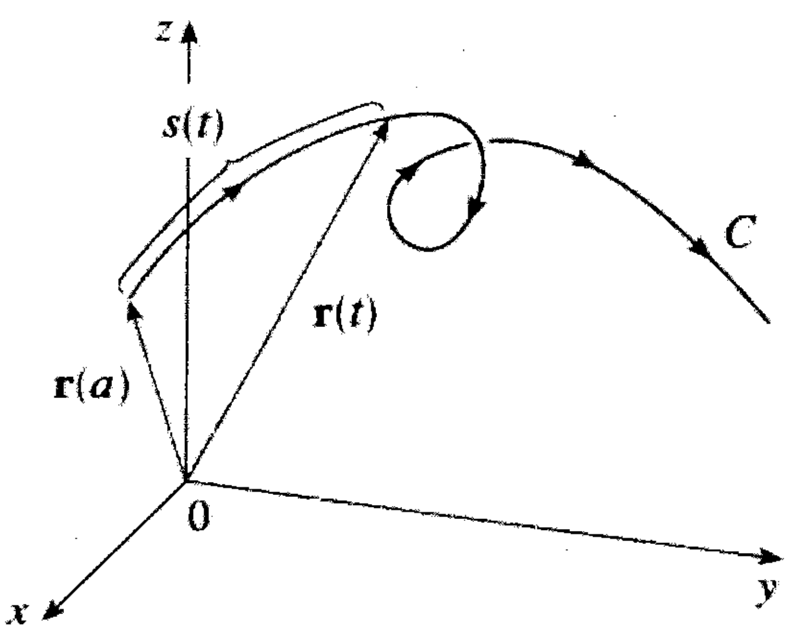
Arc Length Function

The distance from 0 to t is called the *arc length function*

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(u)|du = \text{distance}$$

Note:

$$\frac{ds}{dt} = |\vec{\boldsymbol{r}}'(t)| = \text{speed}$$



Example: x = 3 + 2t, y = 4 – 5t

(a) Find the arc length (from 0 to t).

(b) ***Reparameterize*** in terms of *s(t)*.

$$s = \int_0^t \sqrt{(2)^2 + (-5)^2}\,du$$

$$= \int_0^t \sqrt{29}\,du$$

$$= \sqrt{29}\,u\Big|_0^t$$

SPEED

$$\Rightarrow \boxed{s = \sqrt{29}\,t}$$

DISTANCE TIME

THUS, $t = \frac{1}{\sqrt{29}}s$

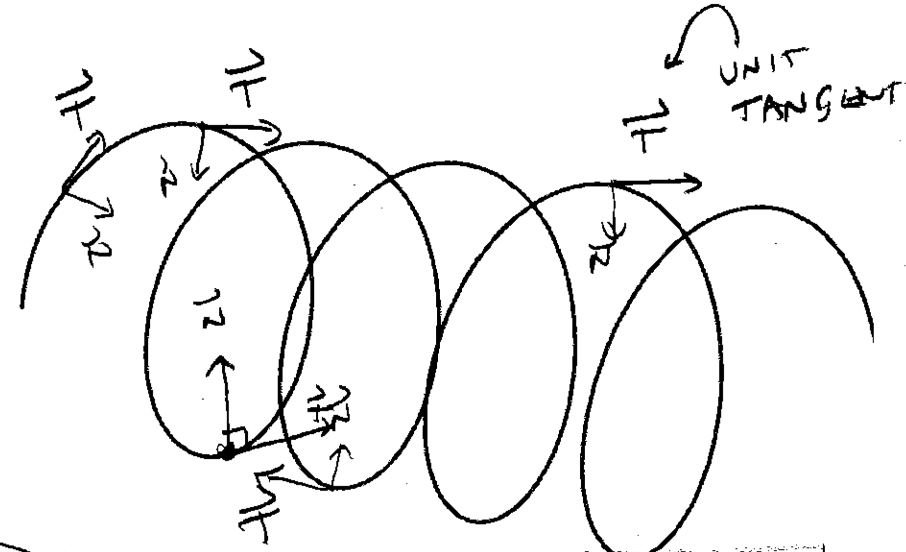
$$x = 3 + 2t = 3 + \frac{2}{\sqrt{29}}s$$

$$y = \underbrace{4 - 5t}_{\text{IN TERMS OF TIME}} = \underbrace{4 - \frac{5}{\sqrt{29}}s}_{\text{IN TERMS OF DIST.}}$$

Unit Tangent & Principal Unit Normal

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$



AT A "LOW" POINT

Example:

$\vec{\boldsymbol{r}}(t) = < 2\sin(3t), t, 2\cos(3t) >$

Find $\vec{\boldsymbol{T}}(\pi)$ and $\vec{\boldsymbol{N}}(\pi)$

$$\boxed{\begin{aligned}\vec{T}(\pi) &= \left< -\frac{6}{\sqrt{37}}, \frac{1}{\sqrt{37}}, 0 \right> \\ \vec{N}(\pi) &= \left< 0, 0, 1 \right>\end{aligned}}$$

$$\vec{r}\,'(t) = \left< 6\cos(3t), 1, -6\sin(3t) \right>$$

$$|\vec{r}\,'(t)| = \sqrt{36\cos^2(3t) + 1^2 + 36\sin^2(3t)} = \sqrt{37}$$

$$\vec{T}(t) = \frac{1}{\sqrt{37}}\vec{r}\,'(t) = \left< \frac{6}{\sqrt{37}}\cos(3t), \frac{1}{\sqrt{37}}, \frac{-6}{\sqrt{37}}\sin(3t) \right>$$

$$\vec{T}\,'(t) = \left< -\frac{18}{\sqrt{37}}\sin(3t), 0, -\frac{18}{\sqrt{37}}\cos(3t) \right>$$

$$|\vec{T}\,'(t)| = \sqrt{\left(\frac{18}{\sqrt{37}}\right)^2\left(\sin^2(3t) + \cos^2(3t)\right)} = \frac{18}{\sqrt{37}}$$

$$\vec{N}(t) = \frac{1}{18/\sqrt{37}}\vec{T}\,'(t) = \left< -\sin(3t), 0, -\cos(3t) \right>$$

Why does this work?

$\boldsymbol{T}$ and $\boldsymbol{T}'$ are always orthogonal.

Proof:

Since $\boldsymbol{T} \cdot \boldsymbol{T} = |\boldsymbol{T}|^2 = 1$,

we can differentiate both sides to get

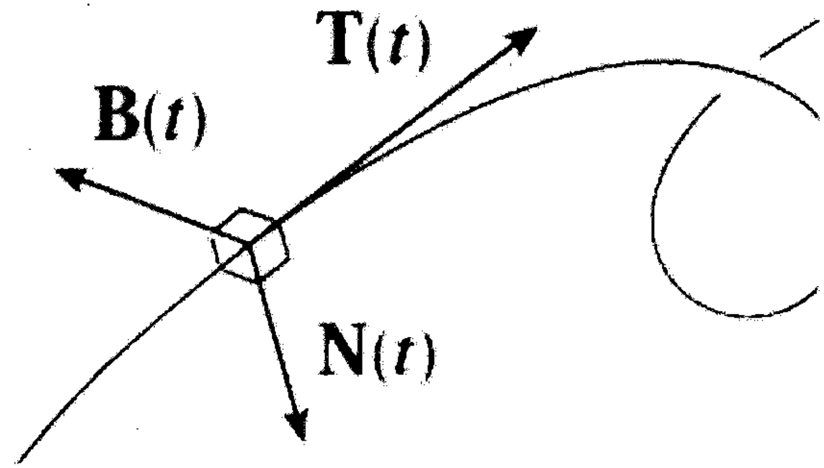
$$\boldsymbol{T}' \cdot \boldsymbol{T} + \boldsymbol{T} \cdot \boldsymbol{T}' = 0.$$

So $2\boldsymbol{T} \cdot \boldsymbol{T}' = 0$.

Thus, $\boldsymbol{T} \cdot \boldsymbol{T}' = 0$. (QED)

Some TNB-Frame Facts:

- $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ point in the tangent and *inward* directions, respectively. Together they give a good approximation of the "plane of motion". This "plane of motion" that goes through a point on the curve and is parallel to $\vec{\boldsymbol{T}}(t)$ and $\vec{\boldsymbol{N}}(t)$ is called the *osculating (kissing) plane*.



- $\vec{\boldsymbol{T}}(t), \vec{\boldsymbol{N}}(t), \vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$ are ALL parallel to the osculating plane. We also define
$\vec{\boldsymbol{B}}(t) = \vec{\boldsymbol{T}}(t) \times \vec{\boldsymbol{N}}(t) = \text{binormal}$
which is orthogonal to all of $\vec{\boldsymbol{T}}(t), \vec{\boldsymbol{N}}(t), \vec{\boldsymbol{r}}'(t)$, and $\vec{\boldsymbol{r}}''(t)$.

Curvature

The **curvature** at a point, *K*, is a measure of how quickly a curve is changing direction at that point.

That is, we define

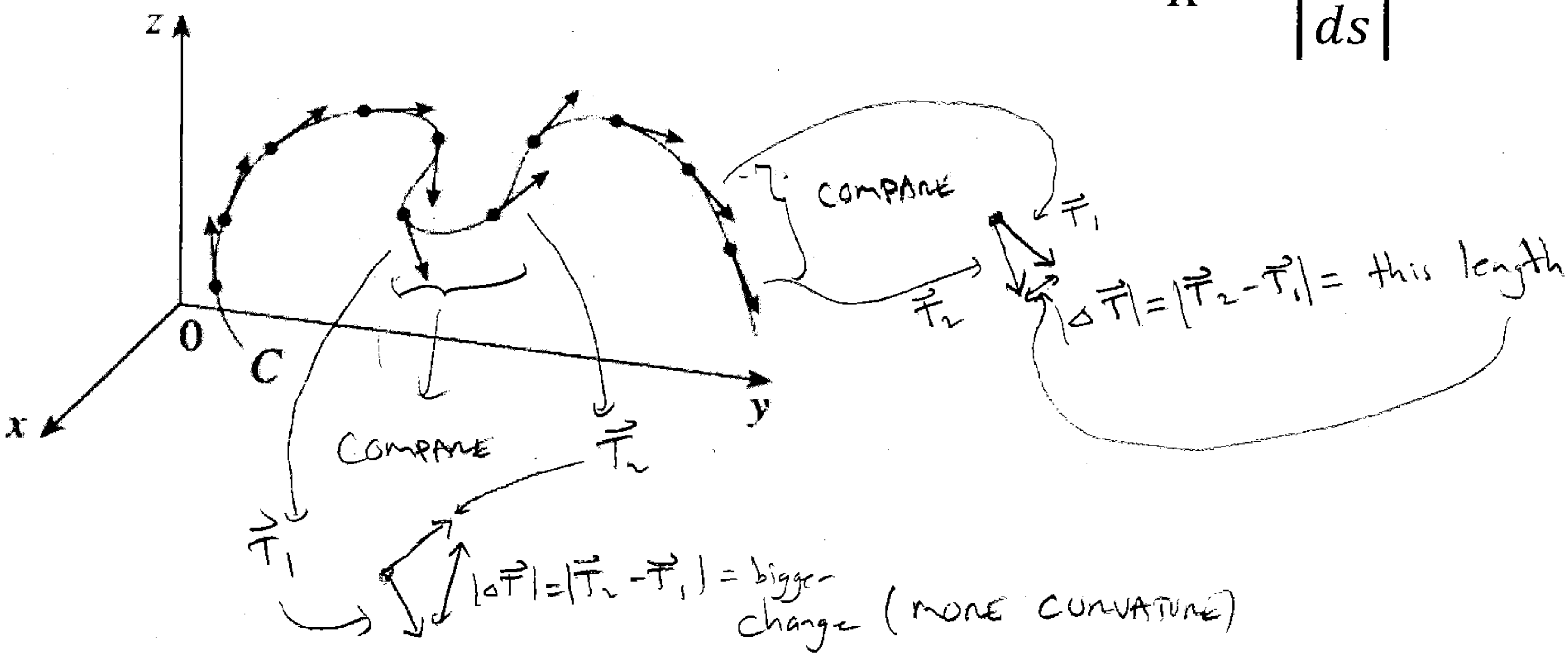
$$K = \frac{change\ in\ direction}{change\ in\ distance}$$

Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$K \approx \left|\frac{\vec{T_2}-\vec{T_1}}{\text{"one inch"}}\right| = \left|\frac{\Delta\vec{T}}{\Delta s}\right|$$

So we define:

$$K = \left|\frac{d\vec{T}}{ds}\right|$$



Computation

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|$$

is not easy to compute directly, so we derive some *shortcuts*

FROM MATH 124

1st shortcut:

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\vec{\boldsymbol{T}}/dt}{ds/dt}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

2nd shortcut

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$$

TAKES SOME WORK TO DERIVE. (SEE LATER IN THESE NOTES)

Example: Find the curvature function for $\boldsymbol{r}(t) = \langle t, \cos(2t), \sin(2t)\rangle$.

$$\boldsymbol{r}'(t) = \langle 1, -2\sin(2t), 2\cos(2t)\rangle$$
$$\boldsymbol{r}''(t) = \langle 0, -4\cos(2t), 4\sin(2t)\rangle$$

$$|\,\boldsymbol{r}'(t)| = \sqrt{1 + 4\sin^2(2t) + 4\cos^2(2t)}$$
so $|\,\boldsymbol{r}'(t)| = \sqrt{5}$

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \langle -8, -4\sin(2t), -4\cos(2t)\rangle$$
So $|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)| = \sqrt{64 + 16} = \sqrt{80}$

$$\frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3} = \frac{\sqrt{80}}{\sqrt{5}^3} = \sqrt{\frac{80}{125}} = 0.8$$

This curve has constant curvature!

Proof of shortcut:

Theorem: $\frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$

Proof:

Since $\boldsymbol{T}(t) = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}$, we have

$$\boldsymbol{r}'(t) = |\boldsymbol{r}'(t)|\boldsymbol{T}(t).$$

Differentiating this gives (prod. rule):

$$\boldsymbol{r}''(t) = |\boldsymbol{r}'(t)|'\boldsymbol{T}(t) + |\boldsymbol{r}'(t)|\boldsymbol{T}'(t).$$

Take cross-prod. of both sides with $\vec{\boldsymbol{T}}$:

$$\boldsymbol{T}\times\boldsymbol{r}'' = |\boldsymbol{r}'|'\,(\boldsymbol{T}\times\boldsymbol{T}) + |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Since $\boldsymbol{T}\times\boldsymbol{T} = <0,0,0>$ (why?) and $\boldsymbol{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|}$, we have

$$\frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,(\boldsymbol{T}\times\boldsymbol{T}').$$

Taking the magnitude gives (why?)

$$\frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|} = |\boldsymbol{r}'|\,|\boldsymbol{T}\times\boldsymbol{T}'| = |\boldsymbol{r}'|\,|\boldsymbol{T}||\boldsymbol{T}'|sin\left(\frac{\pi}{2}\right),$$

Since $|\boldsymbol{T}| = 1$, we have

$$|\boldsymbol{T}'| = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^2}$$

Therefore

$$K = \left|\frac{dT}{ds}\right| = \frac{|\boldsymbol{T}'(t)|}{|\boldsymbol{r}'(t)|} = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3}.$$

Note: To find curvature for a 2D function, $y = f(x)$, we can form a 3D vector function as follows

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle$$

so $\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$ and

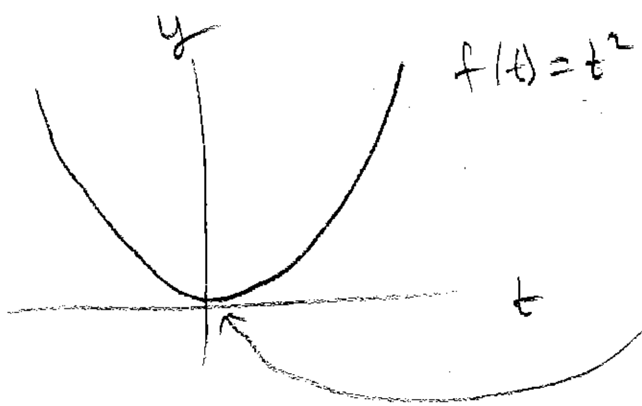
$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\boldsymbol{r}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$



MAXIMUM CURVATURE!

Example: $f(t) = t^2$

Consider $x = t, y = t^2, z = 0$.

At what point (x, y, z) is the curvature maximum?

$$f'(t) = 2t$$

$$f''(t) = 2$$

$$K = \frac{2}{(1+(2t)^2)^{3/2}}$$

$$\boxed{k(t) = \frac{2}{(1+4t^2)^{3/2}}} = 2(1+4t^2)^{-3/2}$$

$$k'(t) = -3(1+4t^2)^{-5/2} \cdot 8t \overset{?}{=} 0 \qquad \boxed{t=0}$$

Summary of 3D Curve Measurement Tools:

Given $\vec{\boldsymbol{r}}(t) = < x(t), y(t), z(t) >$

$\vec{\boldsymbol{r}}'(t) =$ a tangent vector

$$s(t) = \int_0^t |\vec{\boldsymbol{r}}'(t)|dt$$

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

$$\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|} = \text{unit tangent}$$

$$\vec{\boldsymbol{N}}(t) = \frac{\vec{\boldsymbol{T}}'(t)}{|\vec{\boldsymbol{T}}'(t)|} = \text{principal unit normal}$$